

A Transport Equation Approach to Calculations of Green functions and HaMiDeW coefficients

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Building on a fundamental insight due to Avramidi, we provide a system of transport equations for determining key fundamental bi-tensors, including derivatives of the world-function, $\sigma(x, x')$, the square root of the Van Vleck determinant, $\Delta^{1/2}(x, x')$, and the tail-term, $V(x, x')$, appearing in the Hadamard form of the Green function. These bi-tensors are central to a broad range of problems from radiation reaction to quantum field theory in curved spacetime and quantum gravity. Their transport equations may be used either in a semi-recursive approach to determining their covariant Taylor series expansions or as the basis of numerical calculations. To illustrate the power of the semi-recursive covariant series approach, we present an implementation in *Mathematica* which computes very high order covariant series expansions of these objects. Using this, a moderate laptop can, for example, calculate the coincidence limit $[a_7(x, x)]$ and $V(x, x')$ to order $(\sigma^\alpha)^{20}$ in a matter of minutes. Results may be output in either a compact notation or in *xTensor* form. We also numerically integrate the transport equations along null geodesics in Nariai and Schwarzschild spacetimes.

I. INTRODUCTION

In a recent paper [1] we presented methods for obtaining coordinate expansions for the (tail part of the) retarded Green function in spherically symmetric spacetimes. By using computer algebra to obtain high order Taylor series (of order $(\Delta x^\alpha)^{50}$), and applying the theory of Padé approximants we were able to obtain accurate expressions in remarkably large regions. Using these expressions, we were able to present the first complete matched expansion calculation of the self-force in a model ‘black hole’ spacetime, the Nariai spacetime [2], and are currently applying the method to Schwarzschild spacetime. Our ultimate goal in this programme is to work in more general spacetimes, especially Kerr spacetime. A key component of the matched expansion approach is knowledge of the Green function for points close together (i.e. in a *quasilocal* region). As we move away from specific symmetry conditions, we can no longer rely on methods based on a special choice of coordinates in the construction of our quasilocal solution and are led instead to consider other techniques such as transport equations and covariant expansion methods.

Covariant methods for calculating the Green function of the wave operator and the corresponding heat kernel, briefly reviewed in Sec. II below, are central to a broad range of problems from radiation reaction to quantum field theory in curved spacetime and quantum gravity. There is an extremely extensive literature on this topic; here we provide only a very brief overview referring the reader to the reviews by Vassilevich [3] and Poisson [4] and references therein for a more complete discussion. These methods have evolved from pioneering work by Hadamard [5] on the classical theory and DeWitt [6, 7] on the quantum theory. The central objects in the Hadamard and DeWitt covariant expansions are geometrical bi-tensor coefficients $a_n^{AB'}(x, x')$ which are commonly called DeWitt or HaMiDeW coefficients in the physics literature. These coefficients are closely related to the short proper-time asymptotic expansion of the heat kernel of an elliptic operator in a Riemannian space and so are commonly called heat kernel coefficients in the mathematics literature. Traditionally most attention has focused on the *diagonal value* of the heat kernel $K^A_A(x, x; s)$, since the coincidence limits $a_n^A_A(x, x)$ play a central role in the classical theory of spectral invariants [8] and in the quantum theory of the effective action and trace anomalies [9]. By contrast, for the quasilocal part of the matched expansion approach to radiation reaction [10, 11] we seek expansions valid for x and x' as far apart as geometrical methods permit.

The classical approach to the calculation of these coefficients in the physics literature was to use a recursive approach developed by DeWitt [7] in the 1960s. Although these recursive methods work well for the first few terms in the expansion [12, 13], and may be implemented in a tensor software package [14], the amount of calculation required to compute subsequent terms quickly becomes prohibitively long, even when implemented as a computer program. An alternative approach, more common in the mathematics literature, is to use pseudo-differential operators and

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invariance theory [8], where a basis of curvature invariants of the appropriate structure is constructed [15] and then their coefficients determined by explicit evaluation in simple spacetimes. However, here too, the size of the basis grows rapidly and there seems little prospect of reaching orders comparable to those we obtained in the highly symmetric configurations previously studied.

An extremely elegant, non-recursive approach to the calculation of HaMiDeW coefficients has been given by Avramidi [16, 17], but as his motivation was to study the effective action in quantum gravity he was primarily interested in the coincidence limit of the HaMiDeW coefficients, while in the self-force problem, as noted above, we require point-separated expressions. In addition, Avramidi introduced his method in the language of quantum mechanics; quite distinct from the language of transport equations, such as the Raychaudhuri equation, more familiar to discussions of geodesics among relativists. In this paper we present Avramidi's approach in the language of transport equations and show that it is ideal for numerical and symbolic computation. In so doing we are building on the work of Décanini and Folacci [18] who wrote many of the equations we present (we indicate below where we deviate from their approach) and implemented them explicitly by hand. However, calculations by hand are long and inevitably prone to error, particularly for higher spin and for higher order terms in the series and are quite impractical for the very high order expansions we would like for radiation reaction calculations. Instead, we use the transport equations as the basis for *Mathematica* code for algebraic calculations and *C* code for numerical calculations. Rather than presenting our results in excessively long equations (our non-canonical expression for $a_7(x, x)$ for a scalar field contains 2069538 terms!), we have provided these codes online [19, 20].

In Sec. II, we provide a brief review of Green functions, bi-tensors and covariant expansions, outlining the relations between the classical and quantum theories.

In Sec. III, we detail the principles that we consider to encapsulate the key insights of the Avramidi approach and use these to write down a set of transport equations for the key bi-tensors of the theory. These provide an adaptation of the Avramidi approach which is ideally suited to implementation on a computer either numerically or symbolically.

In Sec. IV, we describe a semi-recursive approach to solving for covariant expansion and briefly describe our *Mathematica* implementation of it and its interface to the tensor software package *xTensor*.

In Sec. V, we present a numerical implementation of the the transport equation approach to the calculation of $V(x, x')$ along a null geodesic.

Given our motivation in studying the radiation reaction problem we shall phrase all the discussion of this paper in 4-dimensional spacetime. The reader is referred to Decanini and Folacci [18] for a discussion of the corresponding situation in spacetimes of more general (integer) dimension. We do note however that the HaMiDeW coefficients are purely geometric bi-tensors, formally independent of the spacetime dimension.

Throughout this paper, we use units in which $G = c = 1$ and adopt the sign conventions of [21]. We denote symmetrization of indices using brackets (e.g. $(\alpha\beta)$) and exclude indices from symmetrization by surrounding them by vertical bars (e.g. $(\alpha|\beta|\gamma)$). Roman letters are used for free indices and Greek letters for indices summed over all spacetime dimensions.

II. A BRIEF REVIEW OF GREEN FUNCTIONS, BITENSORS AND COVARIANT EXPANSIONS

A. Classical Green functions

We take an arbitrary field $\varphi^A(x)$ where A denotes the spinorial/tensorial index appropriate to the field, and consider wave operators which are second order partial differential operators of the form [17]

$$\mathcal{D}^A_B = \delta^A_B(\square - m^2) + P^A_B \quad (2.1)$$

where $\square = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ and ∇_α is the covariant derivative defined by a connection $\mathcal{A}^A_{B\alpha}$: $\nabla_\alpha\varphi^A = \partial_\alpha\varphi^A + \mathcal{A}^A_{B\alpha}\varphi^B$, m is the mass of the field and $P^A_B(x)$ is a possible potential term.

In the classical theory of wave propagation in curved spacetime, a fundamental object is the retarded Green function, $G_{\text{ret}}^B{}_{C'}(x, x')$. It is a solution of the inhomogeneous wave equation,

$$\mathcal{D}^A_B G_{\text{ret}}^B{}_{C'}(x, x') = -4\pi\delta^A_{C'}\delta(x, x'), \quad (2.2)$$

with support within the past light-cone of the field point. (The factor of 4π is a matter of convention, our choice here is consistent with Ref. [4].) Finding the retarded Green function globally can be extremely hard, however, provided x and x' are sufficiently close (within a causal domain), we can use the Hadamard form for the retarded Green function solution [5, 22], which in 4 spacetime dimensions takes the form

$$G_{\text{ret}}^A{}_{B'}(x, x') = \theta_-(x, x') \{U^A{}_{B'}(x, x')\delta(\sigma(x, x')) - V^A{}_{B'}(x, x')\theta(-\sigma(x, x'))\}, \quad (2.3)$$

where $\theta_-(x, x')$ is analogous to the Heaviside step-function, being 1 when x' is in the causal past of x , and 0 otherwise, $\delta(x, x')$ is the covariant form of the Dirac delta function, $U^{AB'}(x, x')$ and $V^{AB'}(x, x')$ are symmetric bi-spinors/tensors and are regular for $x' \rightarrow x$. The bi-scalar $\sigma(x, x')$ is the Synge [4] world function, which is equal to one half of the squared geodesic distance between x and x' . The first term, involving $U^A_{B'}(x, x')$, in Eq. (2.3) represents the *direct* part of the Green function while the second term, involving $V^A_{B'}(x, x')$, is known as the *tail* part of the Green function. This tail term represents back-scattering off the spacetime geometry and is, for example, responsible for the quasilocal contribution to the self-force.

Within the Hadamard approach, the symmetric bi-scalar $V^{AB'}(x, x')$ is expressed in terms of a formal expansion in increasing powers of σ [18]:

$$V^{AB'}(x, x') = \sum_{r=0}^{\infty} V_r^{AB'}(x, x') \sigma^r(x, x') \quad (2.4)$$

The coefficients $U^{AB'}$ and $V_r^{AB'}$ are determined by imposing the wave equation, using the identity $\sigma_{;\alpha} \sigma^{;\alpha} = 2\sigma = \sigma_{;\alpha'} \sigma^{;\alpha'}$, and setting the coefficient of each *manifest* power of σ^r equal to zero. Since $V^A_{B'}$ is symmetric for self-adjoint wave operators we are free to apply the wave equation either at x or at x' ; here we choose to apply it at x' . We find that $U^{AB'}(x, x') = \Delta^{1/2}(x, x') g^{AB'}(x, x')$, where $\Delta(x, x')$ is the Van Vleck-Morette determinant defined as [4]

$$\Delta(x, x') = -[-g(x)]^{-1/2} \det(-\sigma_{;\alpha\beta'}(x, x')) [-g(x')]^{-1/2} = \det(-g^{\alpha'}_{\alpha}(x, x') \sigma^{;\alpha}_{\nu'}(x, x')) \quad (2.5)$$

with $g^{\alpha'}_{\alpha}(x, x')$ being the bi-vector of parallel transport (defined fully below). In making this identification we have used the transport equation for the Van Vleck-Morette determinant:

$$\sigma^{;\alpha} \nabla_{\alpha} \ln \Delta = (4 - \square \sigma). \quad (2.6)$$

The coefficients $V_r^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{;\alpha'} (\Delta^{-1/2} V_r^{AB'})_{;\alpha'} + (r+1) \Delta^{-1/2} V_r^{AB'} + \frac{1}{2r} \Delta^{-1/2} \mathcal{D}^{B'}_{C'} V_{r-1}^{AC'} = 0 \quad (2.7a)$$

for $r \in \mathbb{N}$ along with the ‘initial condition’

$$\sigma^{;\alpha'} (\Delta^{-1/2} V_0^{AB'})_{;\alpha'} + \Delta^{-1/2} V_0^{AB'} + \frac{1}{2} \Delta^{-1/2} \mathcal{D}^{B'}_{C'} (\Delta^{1/2} g^{AC'}) = 0. \quad (2.7b)$$

These are transport equations which may be solved in principle within a causal domain by direct integration along the geodesic from x to x' . The complication is that the calculation of $V_r^{AB'}$ requires the calculation of second derivatives of $V_{r-1}^{AB'}$ off the geodesic; we address this issue below.

Finally we note that the Hadamard expansion is an ansatz not a Taylor series. For example, in deSitter spacetime for a conformally invariant scalar theory all the V_r ’s are non-zero while $V \equiv 0$.

B. The quantum theory

In curved spacetime a fundamental object of interest is the Feynman Green function defined for a quantum field $\hat{\varphi}^A(x)$ in the state $|\Psi\rangle$ by

$$G_f^{AB'}(x, x') = iT \left[\langle \Psi | \hat{\varphi}^A(x) \hat{\varphi}^{B'}(x') | \Psi \rangle \right].$$

where T denotes time-ordering. The Feynman Green function may be related to the advanced and retarded Green functions of the classical theory by the covariant commutation relations [7]

$$G_f^{AB'}(x, x') = \frac{1}{8\pi} \left(G_{\text{adv}}^{AB'}(x, x') + G_{\text{ret}}^{AB'}(x, x') \right) + \frac{i}{2} \langle \Psi | \hat{\varphi}^A(x) \hat{\varphi}^{B'}(x') + \hat{\varphi}^{B'}(x') \hat{\varphi}^A(x) | \Psi \rangle.$$

The anticommutator function $\langle \Psi | \hat{\varphi}^A(x) \hat{\varphi}^{B'}(x') + \hat{\varphi}^{B'}(x') \hat{\varphi}^A(x) | \Psi \rangle$ clearly satisfies the homogeneous wave equation so that the Feynman Green function satisfies the equation

$$\mathcal{D}^A_B G_f^{B C'}(x, x') = -\delta^A_{C'} \delta(x, x').$$

Using the proper-time formalism [7], the identity

$$i \int_0^\infty ds e^{-\epsilon s} \exp(isx) = -\frac{1}{x + i\epsilon}, \quad (\epsilon > 0),$$

allows the causal properties of the Feynman function to be encapsulated in the formal expression

$$G_f^A{}_{C'}(x, x') = i \int_0^\infty ds e^{-\epsilon s} \exp(is\mathcal{D})^A{}_B \delta^B{}_{C'} \delta(x, x')$$

where the limit $\epsilon \rightarrow 0+$ is understood. The integrand

$$K^A{}_{C'}(x, x'; s) = \exp(is\mathcal{D})^A{}_B \delta^B{}_{C'} \delta(x, x') \quad (2.8)$$

clearly satisfies the Schrödinger/heat equation

$$\frac{1}{i} \frac{\partial K^A{}_{C'}}{\partial s}(x, x'; s) = \mathcal{D}^A{}_B K^B{}_{C'}(x, x'; s) \quad (2.9)$$

together with the initial condition $K^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'}(x, x')$. The trivial way in which the mass m enters these equations allows it to be eliminated through the prescription

$$K^A{}_{C'}(x, x'; s) = e^{-im^2 s} K_0^A{}_{C'}(x, x'; s), \quad (2.10)$$

with the massless heat kernel satisfying the equation

$$\frac{1}{i} \frac{\partial K_0^A{}_{C'}}{\partial s}(x, x'; s) = (\delta^A{}_B \square + P^A{}_B) K_0^B{}_{C'}(x, x'; s) \quad (2.11)$$

together with the ‘initial condition’ $K_0^A{}_{B'}(x, x'; 0) = \delta^A{}_{B'}(x, x')$.

In 4-dimensional Minkowski spacetime without potential, the massless heat kernel is readily obtained as

$$K_0^A{}_{B'}(x, x'; s) = \frac{1}{(4\pi s)^2} \exp\left(-\frac{\sigma}{2is}\right) \delta^A{}_{B'} \quad (\text{flat spacetime}). \quad (2.12)$$

This motivates the ansatz [7] that in general the massless heat kernel allows the representation

$$K_0^A{}_{B'}(x, x'; s) \sim \frac{1}{(4\pi s)^2} \exp\left(-\frac{\sigma}{2is}\right) \Delta^{1/2}(x, x') \Omega^A{}_{B'}(x, x'; s), \quad (2.13)$$

where $\Omega^A{}_{B'}(x, x'; s)$ possesses the following asymptotic expansion as $s \rightarrow 0+$:

$$\Omega^A{}_{B'}(x, x'; s) \sim \sum_{r=0}^{\infty} a_r^A{}_{B'}(x, x') (is)^r, \quad (2.14)$$

with $a_0^A{}_{B'}(x, x) = \delta^A{}_{B'}$ and $a_r^A{}_{B'}(x, x')$ has dimension $(\text{length})^{-2r}$. The inclusion of the explicit factor of $\Delta^{1/2}$ is simply a matter of convention; by including it we are following DeWitt, but many authors, including Décanini and Folacci, choose instead to include it in the series coefficients

$$A_r^A{}_{B'}(x, x') = \Delta^{1/2} a_r^A{}_{B'}(x, x'). \quad (2.15)$$

It is clearly trivial to convert between the two conventions and, in any case, the coincidence limits agree.

Now, requiring our expansion to satisfy Eq. (2.11) and using the symmetry of $\Omega^A{}_{B'}(x, x'; s)$ to allow operators to act at x' , we find that $\Omega^A{}_{B'}(x, x'; s)$ must satisfy

$$\frac{1}{i} \frac{\partial \Omega^{AB'}}{\partial s} + \frac{1}{is} \sigma^{;\alpha'} \Omega^{AB'}{}_{;\alpha'} = \Delta^{-1/2} (\delta^{B'}{}_{C'} \square + P^{B'}{}_{C'}) \left(\Delta^{1/2} \Omega^{AC'}(x, x'; s) \right).$$

Inserting the expansion Eq. (2.14), the coefficients $a_n^{AB'}(x, x')$ satisfy the recursion relations

$$\sigma^{;\alpha'} a_{r+1}^{AB'}{}_{;\alpha'} + (r+1) a_{r+1}^{AB'} - \Delta^{-1/2} (\delta^{B'}{}_{C'} \square + P^{B'}{}_{C'}) \left(\Delta^{1/2} a_r^{AC'} \right) = 0 \quad (2.16a)$$

for $n \in \mathbb{N}$ along with the ‘initial condition’

$$\sigma^{;\alpha'} a_0^{AB'}{}_{;\alpha'} = 0, \quad (2.16b)$$

with the implicit requirement that they be regular as $x' \rightarrow x$.

Comparing (2.7) and (2.16), one can see that the Hadamard and (mass-independent) HaMiDeW coefficients are related for a theory of mass m by

$$V_r^A{}_{B'}(x, x') = \frac{\Delta^{1/2}(x, x')}{2^{r+1}r!} \sum_{k=0}^{r+1} (-1)^k \frac{(m^2)^{r-k+1}}{(r-k+1)!} a_k^A{}_{B'}(x, x')$$

with inverse

$$a_{r+1}^A{}_{B'}(x, x') = \Delta^{-1/2} \sum_{k=0}^r (-2)^{k+1} \frac{k!}{(r-k)!} (m^2)^{r-k} V_k^A{}_{B'}(x, x') + \frac{(m^2)^{r+1}}{(r+1)!}.$$

In particular,

$$V_r^{(m^2=0)A}{}_{B'}(x, x') = \frac{\Delta^{1/2}(x, x')}{2^{r+1}r!} (-1)^{r+1} a_{r+1}^A{}_{B'}(x, x').$$

These relations enable us to relate the ‘tail term’ of the massive theory to that of massless theory by

$$V(x, x')^A{}_{B'} = \sum_{r=0}^{\infty} V_r^{(m^2=0)A}{}_{B'}(x, x') \frac{(2\sigma)^r r! J_r((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{r/2}} + m^2 \Delta^{1/2} \frac{J_1((-2m^2\sigma)^{1/2})}{(-2m^2\sigma)^{1/2}} \delta^A{}_{B'}.$$

where $J_r(x)$ are Bessel functions of the first kind.

C. Classical Approach to Covariant Expansion Calculations

The Synge world-function, $\sigma(x, x')$ is a bi-scalar, i.e. a scalar at x and at x' defined to be equal to half the square of the geodesic distance between the two points. The world-function may be defined by the fundamental identity

$$\sigma_a \sigma^\alpha = 2\sigma = \sigma_{\alpha'} \sigma^{\alpha'}, \quad (2.17)$$

together with the boundary condition $\lim_{x' \rightarrow x} \sigma(x, x') = 0$ and $\lim_{x' \rightarrow x} \sigma_{ab}(x, x') = g_{ab}(x)$. Here we indicate derivatives at the (un-)primed point by (un-)primed indices:

$$\sigma^a \equiv \nabla^a \sigma \quad \sigma_a \equiv \nabla_a \sigma \quad \sigma^{a'} \equiv \nabla^{a'} \sigma \quad \sigma_{a'} \equiv \nabla_{a'} \sigma. \quad (2.18)$$

σ^a is a vector at x of length equal to the geodesic distance between x and x' , tangent to the geodesic at x and oriented in the direction $x' \rightarrow x$ while $\sigma^{a'}$ is a vector at x' of length equal to the geodesic distance between x and x' , tangent to the geodesic at x and oriented in the opposite direction.

The covariant derivatives of σ may be written as

$$\sigma^a(x, x') = (s - s') u^a \quad \sigma^{a'}(x, x') = (s' - s) u^{a'} \quad (2.19)$$

where s is an affine parameter and u^a is tangent to the geodesic. For time-like geodesics, s may be taken as the proper time along the geodesic while u^a is the 4-velocity of the particle and

$$\sigma(x, x') = -\frac{1}{2}(s - s')^2. \quad (2.20)$$

For null geodesics, u^a is null and $\sigma(x, x') = 0$.

Another bi-tensor of frequent interest is the bi-vector of parallel transport, $g_{ab'}$ defined by the equation

$$g_{ab';\alpha} \sigma^\alpha = 0 = g_{ab';\alpha'} \sigma^{\alpha'} \quad (2.21)$$

with initial condition $\lim_{x' \rightarrow x} g_{ab'}(x, x') = g_{ab}(x)$. From the definition of a geodesic it follows that

$$g_{a\alpha'}\sigma^{\alpha'} = -\sigma_a \quad \text{and} \quad g_{\alpha a'}\sigma^{\alpha} = -\sigma_{a'} \quad (2.22)$$

Given a bi-tensor T_a at x , the parallel displacement bi-vector allows us write T_a as a bi-tensor at x' , obtained by parallel transporting T_a along the geodesic from x to x' and vice-versa,

$$T_\alpha g^{\alpha}_{a'} = T_{a'} \quad T_{\alpha'} g^{\alpha'}_a = T_a. \quad (2.23)$$

Any bi-tensor $T_{a_1 \dots a_m a'_1 \dots a'_n}$ may be expanded in a local covariant Taylor series about the point x :

$$T_{a_1 \dots a_m a'_1 \dots a'_n}(x, x') = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_{a_1 \dots a_m a'_1 \dots a'_n \alpha_1 \dots \alpha_k}(x) \sigma^{\alpha_1} \dots \sigma^{\alpha_k} \quad (2.24)$$

where the $t_{a_1 \dots a_m a'_1 \dots a'_n \alpha_1 \dots \alpha_k}$ are the coefficients of the series and are local tensors at x . Similarly, we can also expand about x' :

$$T_{a_1 \dots a_m a'_1 \dots a'_n}(x, x') = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_{a_1 \dots a_m a'_1 \dots a'_n \alpha'_1 \dots \alpha'_k}(x') \sigma^{\alpha'_1} \dots \sigma^{\alpha'_k} \quad (2.25)$$

For many fundamental bi-tensors, one would typically use the DeWitt approach to determine the coefficients in these expansions as follows:

1. Take covariant derivatives of the defining equation for the bi-tensor (the number of derivatives required depends on the order of the term to be found).
2. Replace all known terms with their coincidence limit, $x \rightarrow x'$.
3. Sort covariant derivatives, introducing Riemann tensor terms in the process.
4. Take the coincidence limit $x' \rightarrow x$ of the result.

This method allows all coefficients to be determined recursively in terms of lower order coefficients and Riemann tensor polynomials. Although this method proves effective for determining the lowest few order terms by hand and can be readily implemented in software, it does not scale well and it is not long before the computation time required to calculate the next term is prohibitively large. It is therefore desirable to find an alternative approach which is more efficient and better suited to implementation in software. In the next section, we will describe one such approach which proves to be highly efficient.

III. AVRAMIDI APPROACH TO COVARIANT EXPANSION CALCULATIONS

Avramidi [16] has proposed an alternative, non-recursive method to the calculation of the coefficients of covariant expansions of fundamental bi-tensors. Translated into our language, this approach emphasizes two fundamental principles when doing calculations:

1. When expanding about x , always try to take derivatives at x' . The result is that derivatives only act on the $\sigma^{a'}$'s and not on the coefficients.
2. Where possible, whenever taking a covariant derivative, $\nabla_{a'}$, contract the derivative with $\sigma^{a'}$.

Applying these two principles, Avramidi has derived non-recursive expressions for the coefficients of covariant expansions of several bi-tensors. As Avramidi's derivations use a rather abstract notation, we will now briefly review his technique in a more explicit notation. We will also extend the derivation to include several other bi-tensors and note that equations (3.11), (3.13), (3.15), (3.17), (3.34), (3.35), (3.46) and (3.49) were previously written down and used by Decanini and Folacci [18].

Throughout this section, we fix the base point x and allow it to be connected to any other point x' by a geodesic. In all cases, we expand about the fixed point, x .

Defining the transport operators D and D' as

$$D \equiv \sigma^{\alpha} \nabla_{\alpha} \quad D' \equiv \sigma^{\alpha'} \nabla_{\alpha'}, \quad (3.1)$$

we can rewrite Eq. (2.17) as

$$(D - 2)\sigma = 0 \qquad (D' - 2)\sigma = 0. \quad (3.2)$$

Differentiating the second of these equations at x and at x' , we get

$$(D' - 1)\sigma^a = 0 \qquad (D' - 1)\sigma^{a'} = 0, \quad (3.3)$$

which, defining

$$\eta^a_{b'} \equiv \sigma^a_{b'} \qquad \xi^{a'}_{b'} \equiv \sigma^{a'}_{b'} \quad (3.4)$$

can be rewritten as

$$\sigma^a = \eta^a_{\alpha'} \sigma^{\alpha'} \qquad \sigma^{a'} = \xi^{a'}_{\alpha'} \sigma^{\alpha'}. \quad (3.5)$$

Finally, we define $\gamma^{a'}_b$, the inverse of $\eta^a_{b'}$,

$$\gamma^{a'}_b \equiv (\eta^b_{a'})^{-1}. \quad (3.6)$$

and also introduce the definition

$$\lambda^a_b \equiv \sigma^a_b. \quad (3.7)$$

We will now derive transport equations for each of these newly introduced quantities along with some others which will be defined as required. Many of these derivations involve considerable index manipulations and are most easily (and accurately) done using a tensor software package such as *xTensor* [23].

The transport equations of this section may be derived in a recursive manner, making use of the identities

$$D'(\sigma_{a'_1 \dots a'_n a'_{n+1}}) = \nabla_{a'_{n+1}}(D'\sigma_{a'_1 \dots a'_n}) - \xi^{\alpha'}_{a'_{n+1}} \nabla_{\alpha'} \sigma_{a'_1 \dots a'_n} + \sigma^{\alpha'} R^{c'}_{a'_1 a'_{n+1} \alpha' \sigma_{c' \dots a'_n}} + \dots + \sigma^{\alpha'} R^{c'}_{a'_n a'_{n+1} \alpha' \sigma_{a'_1 \dots c'}} \quad (3.8)$$

$$D'(\sigma^b_{a'_1 \dots a'_n}) = \nabla^b(D'\sigma_{a'_1 \dots a'_n}) - \eta^b_{\alpha'} \nabla^{\alpha'} \sigma_{a'_1 \dots a'_n}. \quad (3.9)$$

and its generalisation, given below. This method is particularly algorithmic and well suited to implementation on a computer, thus allowing for the automated derivation of a transport equation for an arbitrary number of derivatives of a bi-tensor.

A. Transport equation for $\xi^{a'}_{b'}$

Taking a primed derivative of the second equation in (3.5), we get

$$\xi^{a'}_{b'} = \xi^{a'}_{\alpha' b'} \sigma^{\alpha'} + \xi^{a'}_{\alpha'} \xi^{\alpha'}_{b'}. \quad (3.10)$$

We now commute the last two covariant derivatives in the first term on the right hand side of this equation and rearrange to get:

$$D' \xi^{a'}_{b'} + \xi^{a'}_{\alpha'} \xi^{\alpha'}_{b'} - \xi^{a'}_{b'} + R^{a'}_{\alpha' b' \beta'} \sigma^{\alpha'} \sigma^{\beta'} = 0 \quad (3.11)$$

B. Transport equation for $\eta^a_{b'}$

Taking a primed derivative of the first equation in (3.5), we get

$$\eta^a_{b'} = \eta^a_{\alpha' b'} \sigma^{\alpha'} + \eta^a_{\alpha'} \xi^{\alpha'}_{b'} \quad (3.12)$$

In this case, since σ^a is a scalar at x' , we can commute the two primed covariant derivatives in the first term on the right hand side of this equation without introducing a Riemann term. Rearranging, we get:

$$D' \eta^a_{b'} + \eta^a_{\alpha'} \xi^{\alpha'}_{b'} - \eta^a_{b'} = 0 \quad (3.13)$$

C. Transport equation for $\gamma^{a'}_b$

Solving Eq. (3.13) for $\xi^{a'}_{b'}$ and using (3.6), we get

$$\begin{aligned}\xi^{a'}_{b'} &= \delta^{a'}_{b'} - \gamma^{a'}_{\alpha'} \left(D' \eta^{\alpha'}_{b'} \right) \\ &= \delta^{a'}_{b'} + \left(D' \gamma^{a'}_{\alpha'} \right) \eta^{\alpha'}_{b'},\end{aligned}\quad (3.14)$$

Next, substituting Eq. (3.14) into Eq. (3.11) and rearranging, we get a transport equation for $\gamma^{a'}_{b'}$:

$$(D')^2 \gamma^{a'}_{b'} + D' \gamma^{a'}_{b'} + R^{a'}_{\alpha' \gamma' \beta'} \gamma^{\gamma'}_{b'} \sigma^{\alpha'} \sigma^{\beta'} = 0. \quad (3.15)$$

D. Equation for λ^a_b

Differentiating Eq. (2.17) at x and x' , we get

$$\eta^a_{b'} = \lambda^a_{\alpha} \eta^{\alpha}_{b'} + D \eta^a_{b'} \quad (3.16)$$

which is easily rearranged to give an equation for λ^a_b :

$$\lambda^a_b = \delta^a_b - (D \eta^a_{\alpha'}) \gamma^{\alpha'}_b. \quad (3.17)$$

E. Transport equation for $\sigma^{a'}_{b'c'}$

Applying the identity (3.8) to (3.11) and simplifying the resulting expression, we get

$$\begin{aligned}(D' - 1) \sigma^{a'}_{b'c'} + \sigma^{a'}_{c'} \sigma^a_{\alpha' b'} + \sigma^{\alpha'}_{b'} \sigma^{a'}_{\alpha' c'} + \sigma^{a'}_{\alpha'} \sigma^{\alpha'}_{b' c'} + R^{a'}_{\alpha' b' \beta'; c'} \sigma^{\alpha'} \sigma^{\beta'} \\ - R^{a'}_{\alpha' \beta' b'} \sigma^{\beta'} \sigma^{\alpha'}_{c'} - R^{a'}_{\alpha' \beta' c'} \sigma^{\beta'} \sigma^{\alpha'}_{b'} + R^{a'}_{b' \beta' c'} \sigma^{\beta'} \sigma^{a'}_{\alpha'} = 0\end{aligned}\quad (3.18)$$

F. Transport equation for $\sigma^a_{b'c'}$

Applying the identity (3.9) to (3.11) and simplifying the resulting expression, we get

$$(D' - 1) \sigma^a_{b'c'} + \sigma^{a'}_{b'} \sigma^a_{\alpha' c'} + \sigma^{\alpha'}_{c'} \sigma^a_{\alpha' b'} + \sigma^a_{\alpha'} \sigma^{\alpha'}_{b' c'} + R^{a'}_{b' \beta' c'} \sigma^a_{\alpha'} \sigma^{\beta'} = 0 \quad (3.19)$$

G. Transport equation for $\sigma^{a'}_{b'c'd'}$

Applying the identity (3.8) to (3.18) and simplifying the resulting expression, we get

$$\begin{aligned}(D' - 1) \sigma^{a'}_{b'c'd'} + \sigma^{a'}_{\alpha' b' c'} \sigma^{\alpha'}_{d'} + \sigma^{a'}_{\alpha' b' d'} \sigma^{\alpha'}_{c'} + \sigma^{a'}_{\alpha' c' d'} \sigma^{\alpha'}_{b'} + \sigma^{a'}_{\alpha' b'} \sigma^{\alpha'}_{c' d'} + \sigma^{a'}_{\alpha' c'} \sigma^{\alpha'}_{b' d'} + \sigma^{a'}_{\alpha' d'} \sigma^{\alpha'}_{b' c'} \\ + \sigma^{a'}_{\alpha'} \sigma^{\alpha'}_{b' c' d'} + R^{a'}_{\alpha' \beta' c'} R^{\alpha'}_{d' \gamma' b'} \sigma^{\beta'} \sigma^{\gamma'} + R^{a'}_{\alpha' \beta' b'} R^{\alpha'}_{d' \gamma' c'} \sigma^{\beta'} \sigma^{\gamma'} + R^{a'}_{\alpha' \beta' d'} R^{\alpha'}_{c' \gamma' b'} \sigma^{\beta'} \sigma^{\gamma'} \\ - R^{a'}_{\beta' \alpha' d'} R^{\alpha'}_{b' \gamma' c'} \sigma^{\beta'} \sigma^{\gamma'} - R^{a'}_{\beta' \alpha' c'} R^{\alpha'}_{b' \gamma' d'} \sigma^{\beta'} \sigma^{\gamma'} + R^{a'}_{\beta' b' \gamma'; c' d'} \sigma^{\beta'} \sigma^{\gamma'} + R^{a'}_{b' \beta' c'; d'} \sigma^{\beta'} \sigma^{a'}_{\alpha'} \\ - R^{a'}_{\alpha' \beta' c'; d'} \sigma^{\beta'} \sigma^{\alpha'}_{b'} - R^{a'}_{\alpha' \beta' b'; d'} \sigma^{\beta'} \sigma^{\alpha'}_{c'} - R^{a'}_{\alpha' \beta' b'; c'} \sigma^{\beta'} \sigma^{\alpha'}_{d'} + R^{a'}_{c' \beta' d'} \sigma^{\beta'} \sigma^{a'}_{b' \alpha'} + R^{a'}_{b' \beta' d'} \sigma^{\beta'} \sigma^{a'}_{c' \alpha'} \\ + R^{a'}_{b' \beta' c'} \sigma^{\beta'} \sigma^{a'}_{d' \alpha'} - R^{a'}_{\alpha' \beta' d'} \sigma^{\beta'} \sigma^{\alpha'}_{b' c'} - R^{a'}_{\alpha' \beta' c'} \sigma^{\beta'} \sigma^{\alpha'}_{b' d'} - R^{a'}_{\alpha' \beta' b'} \sigma^{\beta'} \sigma^{\alpha'}_{c' d'} = 0\end{aligned}\quad (3.20)$$

H. Transport equation for $\sigma^a_{b'c'd'}$

Applying the identity (3.9) to (3.18) and simplifying the resulting expression, we get

$$\begin{aligned}(D' - 1) \sigma^a_{b'c'd'} + \sigma^a_{\alpha' b' c'} \sigma^{\alpha'}_{d'} + \sigma^a_{\alpha' b' d'} \sigma^{\alpha'}_{c'} + \sigma^a_{\alpha' c' d'} \sigma^{\alpha'}_{b'} + \sigma^a_{\alpha' b'} \sigma^{\alpha'}_{c' d'} + \sigma^a_{\alpha' c'} \sigma^{\alpha'}_{b' d'} + \sigma^a_{\alpha' d'} \sigma^{\alpha'}_{b' c'} \\ + \sigma^a_{\alpha'} \sigma^{\alpha'}_{b' c' d'} + R^{a'}_{b' \beta' c'; d'} \sigma^{\beta'} \sigma^a_{\alpha'} + R^{a'}_{b' \beta' c'} \sigma^{\beta'} \sigma^a_{d' \alpha'} + R^{a'}_{b' \beta' d'} \sigma^{\beta'} \sigma^a_{c' \alpha'} + R^{a'}_{c' \beta' d'} \sigma^{\beta'} \sigma^a_{b' \alpha'} = 0\end{aligned}\quad (3.21)$$

I. Transport equation for $g_{a'}^b$

The bi-vector of parallel transport is defined by the transport equation

$$D' g_{a'}^b = g_{a'}^b{}_{;\alpha'} \sigma^{\alpha'} \equiv 0. \quad (3.22)$$

J. Transport equation for $g_{ab'}{}_{;c'}$

Let

$$A_{abc} = g_b^{\alpha'} g_c^{\beta'} g_{a\alpha'\beta'} \quad (3.23)$$

Applying D' and commuting covariant derivatives, we get a transport equation for A_{abc} :

$$D' A_{abc} + A_{ab\alpha} \xi^{\beta'}_{\gamma'} g_{\beta'}^{\alpha} g_c^{\gamma'} + g_a^{\alpha'} g_b^{\beta'} g_c^{\gamma'} R_{\alpha'\beta'\gamma'\delta'} \sigma^{\delta'} = 0 \quad (3.24)$$

K. Transport equation for $g_{ab'}{}_{;c}$

Let

$$B_{abc} = g_b^{\beta'} g_{a\beta'}{}_{;c} \quad (3.25)$$

Applying D' and rearranging, we get a transport equation for $B_{\alpha\beta\gamma}$:

$$D' B_{abc} = -A_{ab\alpha} \eta^{\alpha}_{\beta'} g_c^{\beta'} \quad (3.26)$$

L. Transport equation for $g_a^{b'}{}_{;c'd'}$

Applying D' to $g_a^{b'}{}_{;c'd'}$, we get

$$D' g_a^{b'}{}_{;c'd'} = \sigma^{\alpha'} g_a^{b'}{}_{;c'd'\alpha'}. \quad (3.27)$$

Commuting covariant derivatives on the right hand side, this becomes

$$D' g_a^{b'}{}_{;c'd'} = \sigma^{\beta'} \left(g_a^{b'}{}_{;\beta'c'd'} + R^{b'}_{\alpha'\beta'd'} g_a^{\alpha'}{}_{;c'} + R^{b'}_{\alpha'\beta'c'd'} g_a^{\alpha'}{}_{;d'} - R^{\alpha'}_{c'\beta'd'} g_a^{b'}{}_{;\alpha'} + R^{b'}_{\alpha'\beta'c';d'} g_a^{\alpha'} \right). \quad (3.28)$$

Bringing $\sigma^{\beta'}$ inside the derivative in the first time on the right hand side, and noting that $g_a^{b'}{}_{;\beta'} \sigma^{\beta'} = 0$, this then yields a transport equation for $g_a^{b'}{}_{;c'd'}$:

$$\begin{aligned} D' g_a^{b'}{}_{;c'd'} = & -\sigma^{\beta'}_{c'} g_a^{b'}{}_{;\beta'd'} - \sigma^{\beta'}_{d'} g_a^{b'}{}_{;\beta'c'} - \sigma^{\beta'}_{c'd'} g_a^{b'}{}_{;\beta'} \\ & + R^{b'}_{\alpha'\beta'd'} \sigma^{\beta'} g_a^{\alpha'}{}_{;c'} + R^{b'}_{\alpha'\beta'c'} \sigma^{\beta'} g_a^{\alpha'}{}_{;d'} - R^{\alpha'}_{c'\beta'd'} \sigma^{\beta'} g_a^{b'}{}_{;\alpha'} + R^{b'}_{\alpha'\beta'c';d'} \sigma^{\beta'} g_a^{\alpha'}. \end{aligned} \quad (3.29)$$

M. Transport equation for $\zeta = \ln \Delta^{1/2}$

The Van Vleck determinant, Δ is a bi-scalar defined by

$$\Delta(x, x') \equiv \det \left[\Delta^{\alpha'}_{\beta'} \right], \quad \Delta^{\alpha'}_{\beta'} \equiv -g^{\alpha'}_{\alpha} \sigma^{\alpha}_{\beta'} = -g^{\alpha'}_{\alpha} \eta^{\alpha}_{\beta'} \quad (3.30)$$

By Eq. (3.13), we can write the second equation here as:

$$\Delta^{\alpha'}_{\beta'} = -g^{\alpha'}_{\alpha} \left(D' \eta^{\alpha}_{\beta'} + \eta^{\alpha}_{\gamma'} \xi^{\gamma'}_{\beta'} \right) \quad (3.31)$$

Since $D'g^{\alpha'}_{\alpha} = g^{\alpha'}_{\alpha;\beta'}\sigma^{\beta'} = 0$, we can rewrite this as

$$\Delta^{\alpha'}_{\beta'} = D'\Delta^{\alpha'}_{\beta'} + \Delta^{\alpha'}_{\gamma'}\xi^{\gamma'}_{\beta'} \quad (3.32)$$

Introducing the inverse $(\Delta^{-1})^{\alpha'}_{\beta'}$ and multiplying it by the above, we get

$$4 = \xi^{\alpha'}_{\alpha'} + D'(\ln \Delta) \quad (3.33)$$

where we have used the matrix identity $\delta \ln \det \mathbf{M} = \text{Tr} \mathbf{M}^{-1} \delta \mathbf{M}$ to convert the trace to a determinant. This can also be written in terms of $\Delta^{1/2}$:

$$D'(\ln \Delta^{1/2}) = \frac{1}{2} \left(4 - \xi^{\alpha'}_{\alpha'} \right) \quad (3.34)$$

N. Transport equation for the Van Vleck determinant, $\Delta^{1/2}$

By the definition of ζ , the Van Vleck determinant is given by

$$\Delta^{1/2} = e^{\zeta}, \quad (3.35)$$

and so satisfies the transport equation

$$D'\Delta^{1/2} = \frac{1}{2}\Delta^{1/2} \left(4 - \xi^{\alpha'}_{\alpha'} \right). \quad (3.36)$$

O. Equation for $\Delta^{-1/2}D(\Delta^{1/2})$

Defining $\tau = \Delta^{-1/2}D(\Delta^{1/2})$, it is immediately clear that

$$\tau = \Delta^{-1/2}D(\Delta^{1/2}) = D\zeta \quad (3.37)$$

P. Equation for $\Delta^{-1/2}D'(\Delta^{1/2})$

Defining $\tau' = \Delta^{-1/2}D'(\Delta^{1/2})$, it is immediately clear that

$$\tau' = \Delta^{-1/2}D'(\Delta^{1/2}) = D'\zeta \quad (3.38)$$

Q. Equation for $\nabla_{a'}\Delta$

To derive an equation for $\nabla_{a'}\Delta$, we note that

$$\Delta \equiv \det \left[-g^{a'}_{\alpha} \eta^{\alpha}_{b'} \right] = -\det [\eta^a_{b'}] \det \left[g_a^{a'} \right], \quad (3.39)$$

and make use of Jacobi's matrix identity

$$\begin{aligned} d(\det \mathbf{A}) &= \text{tr}(\text{adj}(\mathbf{A}) d\mathbf{A}) \\ &= (\det \mathbf{A}) \text{tr}(\mathbf{A}^{-1} d\mathbf{A}) \end{aligned} \quad (3.40)$$

where the operator d indicates a derivative. Applying (3.40) to (3.39), we get an equation for $\nabla_{a'}\Delta$:

$$\nabla_{a'}\Delta = -\Delta \left[g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha'}{}_{;a'} + \gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'a'} \right] \quad (3.41)$$

R. Equation for $\square'\Delta$

Applying Jacobi's identity twice, together with $d(\mathbf{A}^{-1}) = -\mathbf{A}^{-1}(d\mathbf{A})\mathbf{A}^{-1}$, we find an identity for the second derivative of a matrix:

$$d^2(\det \mathbf{A}) = (\det \mathbf{A}) \left(\text{tr}(\mathbf{A}^{-1}d\mathbf{A}) \text{tr}(\mathbf{A}^{-1}d\mathbf{A}) - \text{tr}(\mathbf{A}^{-1}d\mathbf{A}\mathbf{A}^{-1}d\mathbf{A}) + \text{tr}(\mathbf{A}^{-1}d^2\mathbf{A}) \right). \quad (3.42)$$

Using this identity in Eq. (3.39), we get an equation for $\square'\Delta$,

$$\begin{aligned} \square'\Delta = \Delta \left[\left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha'}{}_{;\mu'} + \gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha';\mu'} + \gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) - \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\beta'}{}_{;\mu'} g_{\beta'}^{\beta} g_{\beta}^{\alpha';\mu'} \right) \right. \\ \left. - \left(\gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\beta'\mu'} \gamma^{\beta'}_{\beta} \sigma^{\beta}_{\alpha'\mu'} \right) + \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha'}{}_{;\mu'} \right) + \left(\gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) \right] \end{aligned} \quad (3.43)$$

S. Equation for $\square'\Delta^{1/2}$

Noting that

$$\square'\Delta^{1/2} = \left(\frac{1}{2} \Delta^{-1/2} \Delta_{;\mu'} \right)^{;\mu'} = \frac{1}{2} \Delta^{-1/2} \square'\Delta - \frac{1}{4} \Delta^{-3/2} \Delta_{;\mu'} \Delta^{;\mu'}, \quad (3.44)$$

it is straightforward to use Eqs. (3.41) and (3.43) to find an equation for $\square'\Delta^{1/2}$:

$$\begin{aligned} \square'\Delta^{1/2} = \frac{1}{2} \Delta^{1/2} \left[\frac{1}{2} \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha'}{}_{;\mu'} + \gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha';\mu'} + \gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) - \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\beta'}{}_{;\mu'} g_{\beta'}^{\beta} g_{\beta}^{\alpha';\mu'} \right) \right. \\ \left. - \left(\gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\beta'\mu'} \gamma^{\beta'}_{\beta} \sigma^{\beta}_{\alpha'\mu'} \right) + \left(g_{\alpha'}^{\alpha} g_{\alpha}^{\alpha'}{}_{;\mu'} \right) + \left(\gamma^{\alpha'}_{\alpha} \sigma^{\alpha}_{\alpha'\mu'} \right) \right] \end{aligned} \quad (3.45)$$

T. Transport equation for V_0

As is given in Eq. (2.7b), V_0 satisfies the transport equation

$$(D' + 1) V_0^{AB'} + \frac{1}{2} V_0^{AB'} \left(\xi^{\mu'}_{\mu'} - 4 \right) + \frac{1}{2} \mathcal{D}^{B'}_{C'} (\Delta^{1/2} g^{AC'}) = 0, \quad (3.46)$$

or equivalently

$$(D' + 1) \left(\Delta^{-1/2} V_0^{AB'} \right) + \frac{1}{2} \Delta^{-1/2} \mathcal{D}^{B'}_{C'} (\Delta^{1/2} g^{AC'}) = 0. \quad (3.47)$$

In particular, for a scalar field:

$$(D' + 1) V_0 + \frac{1}{2} V_0 \left(\xi^{\mu'}_{\mu'} - 4 \right) + \frac{1}{2} (\square' - m^2 - \xi R') \Delta^{1/2} = 0. \quad (3.48)$$

U. Transport equations for V_r

As is given in Eq. (2.7a), V_r satisfies the transport equation

$$(D' + r + 1) V_r^{AB'} + \frac{1}{2} V_r^{AB'} \left(\xi^{\mu'}_{\mu'} - 4 \right) + \frac{1}{2r} \mathcal{D}^{B'}_{C'} V_{r-1}^{AC'} = 0. \quad (3.49)$$

or equivalently

$$(D' + r + 1) \left(\Delta^{-1/2} V_r^{AB'} \right) + \frac{1}{2r} \Delta^{-1/2} \mathcal{D}^{B'}_{C'} V_{r-1}^{AC'} = 0. \quad (3.50)$$

Comparing with Eq. (3.47), it is clear that Eq. (3.50) may be taken to include $r = 0$ if we replace $V_{r-1}^{AC'}/r$ by $\Delta^{1/2} g^{AC'}$.

In particular, for a scalar field:

$$(D' + r + 1) V_r + \frac{1}{2} V_r \left(\xi^{\mu'}_{\mu'} - 4 \right) + \frac{1}{2r} (\square' - m^2 - \xi R') V_{r-1} = 0. \quad (3.51)$$

Together with the earlier equations, the transport equation Eq. (3.46) allows us to immediately solve for $V_0^{AB'}$ along a geodesic. To obtain the higher order V_r we also need to determine $\square' V_{r-1}^{AB'}$. At first sight this appears to require integrating along a family of neighbouring geodesics but, in fact, again we can write transport equations for it. First we note the identity

$$\begin{aligned} \nabla_{a'} (D' T^{AB'}_{a'_1 \dots a'_n}) &= D' (\nabla_{a'} T^{AB'}_{a'_1 \dots a'_n}) + \xi^{\alpha'}_{a'} \nabla_{\alpha'} T^{AB'}_{a'_1 \dots a'_n} + \sigma^{\alpha'} \mathcal{R}^{B'}_{C' a' \alpha'} T^{AC'}_{a'_1 \dots a'_n} \\ &\quad - \sigma^{\alpha'} R^{c'}_{a'_1 \alpha' \alpha'} T^{AC'}_{c' \dots a'_n} - \dots - \sigma^{\alpha'} R^{c'}_{a'_n \alpha' \alpha'} T^{AC'}_{a'_1 \dots c'} \end{aligned} \quad (3.52)$$

where $\mathcal{R}^A_{Bcd} = \partial_c \mathcal{A}^A_{Bd} - \partial_d \mathcal{A}^A_{Bc} + \mathcal{A}^A_{Cd} \mathcal{A}^C_{Bc} - \mathcal{A}^A_{Cd} \mathcal{A}^C_{Bc}$. Working with $\tilde{V}_r^{AB'} = \Delta^{-1/2} V_r^{AB'}$, on differentiating Eq. (3.50) we obtain a transport equation for the first derivative of $V_r^{AB'}$

$$(D' + r + 1) (\tilde{V}_r^{AB'};_{a'}) + \xi^{\alpha'}_{a'} \tilde{V}_r^{AB'};_{\alpha'} + \sigma^{\alpha'} \mathcal{R}^{B'}_{C' a' \alpha'} \tilde{V}_r^{AC'} + \frac{1}{2r} \left(\Delta^{-1/2} \mathcal{D}^{B'}_{C'} \left(\Delta^{1/2} \tilde{V}_{r-1}^{AC'} \right) \right);_{a'} = 0. \quad (3.53)$$

As noted above, this equation also includes $r = 0$ if we replace $\tilde{V}_{r-1}^{AC'}/r$ in this case by $g^{AC'}$:

$$(D' + 1) (\tilde{V}_0^{AB'};_{a'}) + \xi^{\alpha'}_{a'} \tilde{V}_0^{AB'};_{\alpha'} + \sigma^{\alpha'} \mathcal{R}^{B'}_{C' a' \alpha'} \tilde{V}_0^{AC'} + \frac{1}{2} \left(\Delta^{-1/2} \mathcal{D}^{B'}_{C'} \left(\Delta^{1/2} g^{AC'} \right) \right);_{a'} = 0. \quad (3.54)$$

Repeating the process

$$\begin{aligned} (D' + r + 1) (\tilde{V}_r^{AB'};_{a'b'}) &+ \xi^{\alpha'}_{b'} \tilde{V}_r^{AB'};_{\alpha' a'} + \xi^{\alpha'}_{a'} \tilde{V}_r^{AB'};_{\alpha' b'} \\ &+ \sigma^{\alpha'} \mathcal{R}^{B'}_{C' b' \alpha'} \tilde{V}_r^{AC'};_{a'} + \sigma^{\alpha'} \mathcal{R}^{B'}_{C' a' \alpha'} \tilde{V}_r^{AC'};_{b'} + \xi^{\alpha'}_{a';b'} \tilde{V}_r^{AB'};_{\alpha'} - \sigma^{\alpha'} R^{\beta'}_{a' b' \alpha'} \tilde{V}_r^{AC'};_{\beta'} \\ &+ \xi^{\alpha'}_{b'} \mathcal{R}^{B'}_{C' a' \alpha'} \tilde{V}_r^{AC'} + \sigma^{\alpha'} \mathcal{R}^{B'}_{C' a' \alpha';b'} \tilde{V}_r^{AC'} + \frac{1}{2r} \left(\Delta^{-1/2} \mathcal{D}^{B'}_{C'} \left(\Delta^{1/2} \tilde{V}_{r-1}^{AC'} \right) \right);_{a'b'} = 0, \end{aligned} \quad (3.55)$$

with the $\tilde{V}_0^{AB'};_{a'b'}$ equation given by the same replacement as above.

Clearly this process may be repeated as many times as necessary. At each stage we require two more derivatives on $\tilde{V}_{r-1}^{AC'}$ than on $\tilde{V}_r^{AC'}$ but this may be obtained by a bootstrap process grounded by the $\tilde{V}_0^{AC'}$ equation which involves only the fundamental objects $\Delta^{1/2}$ and $g^{AC'}$ which we have explored above. As with our previous equations, while this process quickly becomes too tedious to follow by hand it is straightforward to programme.

For example, to determine V_1 for a scalar field we first need to solve the two transport equations

$$(D' + 1) (\tilde{V}_0;_{a'}) + \xi^{c'}_{a'} \tilde{V}_0;_{c'} + \frac{1}{2} \left(\Delta^{-1/2} (\square' - m^2 - \xi R') \Delta^{1/2} \right);_{a'} = 0, \quad (3.56)$$

and

$$\begin{aligned} (D' + 1) (\tilde{V}_0;_{a'b'}) &+ \xi^{c'}_{b'} \tilde{V}_0;_{a'c'} + \xi^{c'}_{a'} \tilde{V}_0;_{c'b'} + \\ &+ \xi^{c'}_{a';b'} \tilde{V}_0;_{c'} - \sigma^{c'} R^{d'}_{a' b' c'} \tilde{V}_0;_{d'} + \frac{1}{2} \left(\Delta^{-1/2} (\square' - m^2 - \xi R') \Delta^{1/2} \right);_{a'b'} = 0. \end{aligned} \quad (3.57)$$

In the next two sections we show how the above system of transport equations can be solved either as a series expansion or numerically. For sufficiently simple spacetimes, it is also possible to find closed form solutions which provide a useful check on our results.

IV. SEMI-RECURSIVE APPROACH TO COVARIANT EXPANSIONS

In this section, we will investigate solutions to the transport equations of Sec. III in the form of covariant series expansions. The goal is to find covariant series expressions for the HaMiDeW coefficients. Several methods have been previously applied for doing such calculations, both by hand and using computer algebra [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]. However, this effort has been focused primarily on the calculation of

the *diagonal* coefficients. To the our knowledge only the work of Decanini and Folacci [18], upon which our method is based, has been concerned with the off-diagonal coefficients.

Before proceeding further, it is helpful to see how covariant expansions behave under certain operations. First, applying the operator D' to the covariant expansion of any bi-tensor $A_{a_1 \dots a_m a'_1 \dots a'_n}$ about the point x , we get

$$\begin{aligned} D' A_{a_1 \dots a_m a'_1 \dots a'_n}(x, x') &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k}{}_{\alpha'} \sigma^{\alpha'} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} k a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k} \end{aligned} \quad (4.1)$$

where the second line is obtained by applying Eq. (3.5) to the first line. In other words, applying D' to the k -th term in the series is equivalent to multiplying that term by k .

Next we consider applying the operator D to the covariant expansion of any bi-tensor $A_{a_1 \dots a_m a'_1 \dots a'_n}$ about the point x . In this case, there will also be a term involving the derivative of the series coefficient, giving

$$\begin{aligned} D A_{a_1 \dots a_m a'_1 \dots a'_n}(x, x') &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[k a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k}{}_{\alpha} \sigma^{\alpha} + a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k; \alpha}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k} \sigma^{\alpha} \right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[k a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k} + a_{a_1 \dots a_m a'_1 \dots a'_n \beta_1 \dots \beta_k; \alpha}(x) \sigma^{\beta_1} \dots \sigma^{\beta_k} \sigma^{\alpha} \right]. \end{aligned} \quad (4.2)$$

We can also consider multiplication of covariant expansions. For any two tensors, A^a_b and B^a_b , with product $C^a_{\alpha} \equiv A^{\alpha}_c B^c_b$, say, we can relate their covariant expansions by

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} C^a_b{}_{\beta_1 \dots \beta_n} \sigma^{\beta_1} \dots \sigma^{\beta_n} &= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} A^a_{\alpha \beta_1 \dots \beta_n} \sigma^{\beta_1} \dots \sigma^{\beta_n} \right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B^{\alpha}_b{}_{\beta_1 \dots \beta_n} \sigma^{\beta_1} \dots \sigma^{\beta_n} \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k=0}^n \binom{n}{k} A^a_{\alpha \beta_1 \dots \beta_k} B^{\alpha}_b{}_{\beta_{k+1} \dots \beta_n} \sigma^{\beta_1} \dots \sigma^{\beta_n}. \end{aligned} \quad (4.3)$$

Finally, many of the equations derived in the previous section contain terms involving the Riemann tensor at x' , $R^{a'}_{b'c'd'}$. As all other quantities are expanded about x rather than x' , we will also need to rewrite these Riemann terms in terms of their expansion about x :

$$\begin{aligned} R^{a'}_{\alpha' b' \beta' \gamma'} \sigma^{\alpha'} \sigma^{\beta'} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} R^a_{(\alpha | b | \beta; \gamma_1 \dots \gamma_k)} \sigma^{\alpha} \sigma^{\beta} \sigma^{\gamma_1} \dots \sigma^{\gamma_k} \\ &= \sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} \mathcal{K}^a_b{}_{(k)}, \end{aligned} \quad (4.4)$$

where we follow Avramidi [16] in introducing the definition

$$\begin{aligned} \mathcal{K}^a_b{}_{(n)} &\equiv R^a_{(\alpha_1 | b | \alpha_2; \alpha_3 \dots \alpha_n)} \sigma^{\alpha_1} \dots \sigma^{\alpha_n} \\ &\equiv \bar{\mathcal{K}}^a_b{}_{(n)} \sigma^{\alpha_1} \dots \sigma^{\alpha_n}. \end{aligned} \quad (4.5)$$

These four considerations will now allow us to rewrite the transport equations of Sec. III as recursion relations for the coefficients of the covariant expansions of the tensors involved.

A. Recursion relation for coefficients of the covariant expansion of $\gamma^{a'}_b$

Rewriting Eq. (3.15) in terms of covariant expansions, we get

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (k^2 + k) \gamma^{a'}_{b \alpha_1 \dots \alpha_k}(x) \sigma^{\alpha_1} \dots \sigma^{\alpha_k} + \left(\sum_{k=2}^{\infty} \frac{(-1)^k}{(k-2)!} \mathcal{K}^{a'}_{b (k)} \right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma^{a'}_{b \alpha_1 \dots \alpha_k}(x) \sigma^{\alpha_1} \dots \sigma^{\alpha_k} \right) = 0. \quad (4.6)$$

From this, the n -th term in the covariant series expansion of $\gamma^{a'}_b$ can be written recursively in terms of products of lower order terms in the series with \mathcal{K} :

$$\begin{aligned} \gamma^{a'}_{b (0)} &= -\delta^{a'}_b \quad \gamma^{a'}_{\nu (1)} = 0 \\ \gamma^{a'}_{b (n)} &= -\left(\frac{n-1}{n+1} \right) \sum_{k=0}^{n-2} \binom{n-2}{k} g^{a'}_{\alpha} g^{\beta}_{\beta'} \bar{\mathcal{K}}^{\alpha}_{\beta (n-k)} \gamma^{\beta'}_{b (k)}. \end{aligned} \quad (4.7)$$

Many of the following recursion relations will make use of these coefficients.

B. Recursion relation for coefficients of the covariant expansion of $\eta^a_{b'}$

Since $\gamma^{a'}_b$ is the inverse of $\eta^a_{b'}$, we have

$$\gamma^{a'}_{\alpha} \eta^{\alpha}_{b'} = \delta^{a'}_{b'}. \quad (4.8)$$

Substituting in covariant expansion expressions for $\gamma^{a'}_{\alpha}$ and $\eta^{\alpha}_{b'}$, we find that the n -th term in the covariant series expansion of $\eta^a_{b'}$ is

$$\eta^a_{b' (0)} = -\delta^a_{b'} \quad \eta^a_{b' (1)} = 0 \quad \eta^a_{b' (n)} = \sum_{k=2}^n \binom{n}{k} g^a_{\alpha'} \gamma^{\alpha'}_{\beta (k)} \eta^{\beta}_{b' (n-k)}. \quad (4.9)$$

C. Recursion relation for coefficients of the covariant expansion of $\xi^{a'}_{b'}$

Writing Eq. (3.14) in terms of covariant series, it is immediately apparent that the n -th term in the covariant expansion of $\xi^{a'}_{b'}$ is

$$\begin{aligned} \xi^{a'}_{b' (0)} &= \delta^{a'}_{b'} \quad \xi^{a'}_{b' (1)} = 0 \\ \xi^{a'}_{b' (n)} &= n g^{a'}_{\alpha} \eta^{\alpha}_{b' (n)} - \sum_{k=2}^{n-2} \binom{n}{k} k \gamma^{a'}_{\alpha (n-k)} \eta^{\alpha}_{b' (k)}. \end{aligned} \quad (4.10)$$

D. Recursion relation for coefficients of the covariant expansion of λ^a_b

Using Eq. (3.17), we can write an equation for the n -th order coefficient of the covariant expansion of λ^a_b . However, the expression involves the operator D acting on the covariant series expansion of $\eta^a_{b'}$, so we will first need to find an expression for that. As discussed in the beginning of this section, the derivative in D will affect both the coefficient and the σ^a 's. When acting on the σ^a 's, it has the effect of multiplying the term by n as was previously the case with D' . When acting on the coefficient, it will add a derivative to it and increase the order of the term (since we will then be adding a σ^a). We now appeal to the fact that the terms in the expansion of $\eta^a_{b'}$ consists solely of products of $\mathcal{K}^a_{b (k)}$. This means that applying the following general rules when D acts on $\mathcal{K}^a_{b (k)}$, we will get the desired result:

- $D\mathcal{K}^a_{b (k)} = k\mathcal{K}^a_{b (k)} + \mathcal{K}^a_{b (k+1)}$

- When encountering compound expressions (i.e. consisting of more than a single $\mathcal{K}_{b(k)}^a$), use the normal rules for differentiation (product rule, distributivity, etc.)

Taking this into consideration and letting D^+ signify the contribution at one higher order and D^0 signify the contribution that keeps the order the same, we can write the general n -th term in the covariant series expansion of $D\eta_{b'}^a$ as

$$(D\eta_{b'}^a)_{(n)} = D^+\eta_{b'(n-1)}^a - D^0\eta_{b'(n)}^a \quad (4.11)$$

It is then straightforward to write an expression for the n -th term in the covariant series expansion of $\lambda_{b'}^a$:

$$\lambda_{b(n)}^a = \sum_{k=0}^{n-2} \binom{n}{k} (n-k) \left(D^+\eta_{\alpha'(n-k-1)}^a - D^0\eta_{\alpha'(n-k)}^a \right) \gamma_{b(k)}^{\alpha'} \quad (4.12)$$

E. Recursion relation for coefficients of the covariant expansion of A_{abc}

We can rewrite Eq. (3.24) as

$$(D' + 1)(A_{ab\alpha}\gamma_c^\alpha) + R_{ab\alpha\beta}\sigma^\alpha\gamma_c^\beta = 0, \quad (4.13)$$

which when rewritten in terms of covariant series becomes

$$A_{abc(k)} = -\frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} k \mathcal{R}_{ab\alpha(k)} \gamma_{c(n-k)}^\alpha + \sum_{k=0}^{n-2} \binom{n}{k} A_{ab\alpha(k)} \gamma_{c(n-k)}^\alpha \quad (4.14)$$

where we follow Avramidi [16, 17] in defining

$$\mathcal{R}_{abc(n)} \equiv R_{ab(\alpha_1|c|\alpha_2\cdots\alpha_n)} \sigma^{\alpha_1} \cdots \sigma^{\alpha_n} \quad (4.15)$$

Alternatively, writing Eq. (3.24) directly in terms of covariant series, we get

$$A_{abc(k)} = \frac{n}{n+1} \mathcal{R}_{abc(n)} - \frac{1}{n+1} \sum_{k=0}^{n-2} \binom{n}{k} A_{ab\alpha(k)} \xi_{c(n-k)}^\alpha, \quad (4.16)$$

which has the benefit of requiring half as much computation as the previous expression.

F. Recursion relation for coefficients of the covariant expansion of B_{abc}

By Eq. (3.26), we can immediately write an equation for the coefficients of the covariant expansion of B_{abc} :

$$B_{abc(n)} = \frac{1}{n} \sum_{k=0}^n \binom{n}{k} A_{ab\alpha(k)} \eta_{c(n-k)}^\alpha \quad (4.17)$$

G. Covariant expansion of ζ

From Eq. (3.34) we immediately obtain expressions for the coefficients of the covariant series of ζ :

$$\zeta_{(0)} = 0 \quad \zeta_{(1)} = 0 \quad \zeta_{(n)} = -\frac{1}{2n} \xi_{\rho'(n)}^{\rho'} \quad (4.18)$$

H. Recursion relation for $\Delta^{1/2}$

Since $\zeta = \ln \Delta^{1/2}$, we can write

$$\Delta^{1/2} D' \zeta = D' \Delta^{1/2}. \quad (4.19)$$

This allows us to write down a recursive equation for the coefficients of the covariant series expansion of $\Delta^{1/2}$,

$$\Delta_{(n)}^{1/2} = \frac{1}{n} \sum_{k=2}^n \binom{n}{k} k \zeta_{(k)} \Delta_{(n-k)}^{1/2}. \quad (4.20)$$

I. Recursion relation for $\Delta^{-1/2}$

Given the covariant series expansion for $\Delta^{1/2}$, it is straightforward to calculate the covariant expansion of $\Delta^{-1/2}$, which is simply

$$\Delta_{(n)}^{-1/2} = \begin{cases} -\Delta_{(n)}^{1/2} & n \text{ odd,} \\ \Delta_{(n)}^{1/2} & n \text{ even.} \end{cases} \quad (4.21)$$

J. Covariant expansion of τ

Eq. (3.37) may be immediately written as a covariant series equation,

$$\tau_{(n)} = -nD^+\zeta_{(n-1)} + D^0\zeta_{(n)}. \quad (4.22)$$

K. Covariant expansion of τ'

Eq. (3.38) may be immediately written as a covariant series equation,

$$\tau'_{(n)} = n\zeta_{(n)}. \quad (4.23)$$

L. Covariant expansion of covariant derivative at x' of a bi-scalar

Let $T(x, x')$ be a general bi-scalar. Writing T as a covariant series,

$$T = \sum_{n=0}^{\infty} T_{(n)} = \sum_{n=0}^{\infty} T_{\alpha_1 \dots \alpha_n}(x) \sigma^{\alpha_1} \dots \sigma^{\alpha_n}, \quad (4.24)$$

and applying a covariant derivative at x' , we get

$$\begin{aligned} T_{;b'} &= \sum_{n=0}^{\infty} T_{(n);b'} \\ &= \sum_{n=0}^{\infty} n T_{(\alpha_1 \dots \alpha_n)} \sigma^{\alpha_1} \dots \sigma^{\alpha_{n-1}} \sigma^{\alpha_n}_{b'} \\ &= \sum_{n=0}^{\infty} n T_{(\alpha_1 \dots \alpha_{n-1} \rho)} \sigma^{\alpha_1} \dots \sigma^{\alpha_{n-1}} \eta^\rho_{b'} \\ &= - \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} (T_{(k+1)})_{(-1)} \eta_{(n-k)} \end{aligned} \quad (4.25)$$

where we have introduced the notation

$$(T_{(n)})_{(-k)} \equiv T_{(\alpha_1 \dots \alpha_{(n-k)} a_{(n-k+1)} \dots \alpha_n)} \sigma^{\alpha_1} \dots \sigma^{\alpha_{n-k}} \quad (4.26)$$

M. Covariant expansion of d'Alembertian at x' of a bi-scalar

Let $T(x, x')$ be a general bi-scalar as in the previous section. Applying (4.25) twice and taking care to include the term involving $g_a{}^{b'}$, we can then write the d'Alembertian, $\square' T(x, x')$ at x' in terms of covariant series,

$$(\square' T)_{(n)} = - \sum_{k=0}^n \binom{n}{k} (T_{;a(k+1)})_{(-1)} \eta_{(n-k)} - \sum_{k=1}^n \binom{n}{k} T_{(k)} X_{;a(n-k)}. \quad (4.27)$$

N. Covariant expansion of $\nabla_a \Delta^{1/2}$

Applying Eq. (4.25) to the case $T = \Delta^{1/2}$, we get

$$\Delta_{;a}^{1/2} = - \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \left(\Delta_{(k+1)}^{1/2} \right)_{(-1)} \eta_{(n-k)}. \quad (4.28)$$

O. Covariant expansion of $\square' \Delta^{1/2}$

Applying Eq. (4.27) to the case $T = \Delta^{1/2}$, we get

$$(\square' \Delta^{1/2})_{(n)} = - \sum_{k=0}^n \binom{n}{k} \left(\Delta_{;a(k+1)}^{1/2} \right)_{(-1)} \eta_{(n-k)} - \sum_{k=1}^n \binom{n}{k} A_{(k)} \Delta_{;a(n-k)}^{1/2}, \quad (4.29)$$

where $A_{(n)}$ is the n -th term in the covariant series of the tensor defined in (3.24).

P. Covariant expansion of V_0

The transport equation for V_0 , Eq. (3.46), can be written in the alternative form

$$(D' + 1) V_0 - V_0 \tau' + \frac{1}{2} (\square' - m^2 - \xi R) \Delta^{1/2} = 0. \quad (4.30)$$

This equation is then easily written in terms of covariant expansion coefficients,

$$V_{0(n)} = \frac{1}{n+1} \left(\sum_{k=0}^{n-2} \binom{n}{k} V_{0(k)} \tau'_{(n-k)} - \frac{1}{2} \left((\square' \Delta^{1/2})_{(n)} - (m^2 + \xi R) \Delta_{(n)}^{1/2} \right) \right) \quad (4.31)$$

Q. Covariant expansion of V_r

The transport equation for V_r , Eq. (3.49) can also be written in the alternative form

$$(D' + r + 1) V_r - V_r \tau' + \frac{1}{2r} (\square' - m^2 - \xi R) V_{r-1} = 0. \quad (4.32)$$

Again, this is easily written in terms of covariant expansion coefficients,

$$V_{r(n)} = \frac{1}{r+n+1} \left(\sum_{k=0}^{n-2} \binom{n}{k} V_{r(k)} \tau'_{(n-k)} - \frac{1}{2r} \left((\square V_{r-1})_{(n)} - (m^2 + \xi R) V_{r-1(n)} \right) \right). \quad (4.33)$$

Tables I and II illustrate the performance of our implementation on a desktop computer (2.4GHz Quad Core Processor with 8GB RAM).

V. NUMERICAL SOLUTION OF TRANSPORT EQUATIONS

In this section, we describe the implementation of the numerical solution of the transport equations of Sec. III. We use the analytic results for σ , $\Delta^{1/2}$, $g_a{}^{b'}$ and V_0 for a scalar field in Nariai spacetime from Ref. [43] as a check on our numerical code.

For the purposes of numerical calculations, the operator \mathcal{D}' acting on a general bi-tensor $T^{a' \dots}_{b' \dots}$ can be written as

$$\mathcal{D}' T^{a' \dots}_{b' \dots} = (s' - s) \left(\frac{d}{ds} T^{a' \dots}_{b' \dots} + T^{\alpha' \dots}_{b' \dots} \Gamma_{\alpha' \beta'}^{a'} u^{\beta'} + \dots - T^{a' \dots}_{\alpha' \dots} \Gamma_{b' \beta'}^{\alpha'} u^{\beta'} - \dots \right), \quad (5.1)$$

HaMiDeW Coefficient, a_n	Calculation Time (seconds)	Number of terms	Memory Used (bytes)
a_0	0	0	16
a_1	0.004	2	288
a_2	0	7	2 984
a_3	0.016	68	36 976
a_4	0.168	787	522 952
a_5	3.012	10 183	7 993 792
a_6	70.196	141 691	128 298 192
a_7	1016.696	2 069 538	2 123 985 816

TABLE I: Calculation performance of our semi-recursive implementation of the Avramidi method for computing the coincident (diagonal) HaMiDeW coefficients.

Order	Calculation Time (seconds)	Number of terms	Memory Used (bytes)
0	0	2	528
1	0	2	288
2	0	8	3 056
3	0.004	12	4 800
4	0.007	41	18 400
5	0.016	72	34 328
6	0.024	189	96 920
7	0.048	357	193 120
8	0.084	810	464 240
9	0.132	1 568	938 960
10	0.188	3 290	2 067 512
11	0.384	6 350	4 164 256
12	0.692	12 732	8 689 208
13	1.308	24 340	17 230 944
14	2.688	47 291	34 697 312
15	5.156	89 397	67 881 528
16	9.692	169 900	133 241 688
17	19.985	317 417	256 127 816
18	39.582	593 371	494 810 408
19	76.724	1 096 634	937 590 072
20	122.943	2 023 297	1 766 643 856

TABLE II: Calculation performance of the Avramidi method for computing the covariant series expansion of V_0 .

where s is the affine parameter, $\Gamma_{b'c'}^{a'}$ are the Christoffel symbols at x' and $u^{a'}$ is the four velocity at x' . Additionally, we make use of the fact that

$$\sigma^{a'} = (s' - s)u^{a'}. \quad (5.2)$$

which allows us to write Eqs. (3.11), (3.13), (3.18), (3.19), (3.20), (3.21), (3.22), (3.24), (3.29), (3.34), (3.45) and (3.46) as a system of coupled, tensor ordinary differential equations. These equations all have the general form:

$$\frac{d}{ds} T^{a' \dots}_{b' \dots} = (s')^{-1} A^{a' \dots}_{b' \dots} + B^{a' \dots}_{b' \dots} + s' C^{a' \dots}_{b' \dots} - T^{\alpha' \dots}_{b' \dots} \Gamma_{\alpha' \beta'}^{a'} u^{\beta'} - \dots + T^{a' \dots}_{\alpha' \dots} \Gamma_{b' \beta'}^{\alpha'} u^{\beta'} + \dots, \quad (5.3)$$

where we have set $s = 0$ without loss of generality and where $A^{a' \dots}_{b' \dots} = 0$ initially (i.e. at $s' = 0$). It is not necessarily true, however, that the derivative of $A^{a' \dots}_{b' \dots}$ is zero initially. This fact is important when considering initial data for the numerical scheme.

Solving this system of equations along with the geodesic equations for the spacetime of interest will then yield a numerical value for V_0 . Moreover, since $V = V_0$ along a null geodesic, the transport equation for V_0 will effectively give the full value of V on the light-cone. We have implemented this numerical integration scheme for geodesics in Nariai and Schwarzschild spacetimes using the Runge-Kutta-Fehlberg method in GSL [44]. The source code of our implementation is available online [20].

A. Initial Conditions

Numerical integration of the transport equations requires initial conditions for each of the bi-tensors involved. These initial conditions are readily obtained by considering the covariant series expansions of V_0 , $\Delta^{1/2}$, $\xi^{a'}_{b'}$, $\eta^a_{b'}$ and $g_a^{b'}$ and their covariant derivatives at x' . Initial conditions for all bi-tensors used for calculating V_0 are given in Table III, where we list the transport equation for the bi-tensor, the bi-tensor itself and its initial value.

Additionally, as is indicated in Eq. (5.3), many of the transport equations will contain terms involving $(s')^{-1}$. These terms must obviously be treated with care in any numerical implementation. Without loss of generality, we set $s = 0$. Then, for the initial time step ($s' = s$), we require analytic expressions for

$$\lim_{s' \rightarrow s} (s')^{-1} A^{\alpha'}_{b' \dots} \quad (5.4)$$

which are then be used to numerically compute an accurate initial value for the derivative. This limit can be computed from the first order term in the covariant series of $A^{\alpha'}_{b' \dots}$, which is found most easily by considering the covariant series of its constituent bi-tensors. For this reason, we list in Table III the limit as $s' \rightarrow 0$ of all required bi-tensors multiplied by $(s')^{-1}$. In Table IV we list the terms $(s')^{-1} A^{\alpha'}_{b' \dots}$ for each transport equation involving $(s')^{-1}$, along with their limit as $s \rightarrow 0$.

Equation	Bitensor	Initial Condition	$(s')^{-1}$ Initial Condition
(3.11)	$\xi^{a'}_{b'}$	δ^a_b	0
(3.13)	$\eta^a_{b'}$	$-g^a_b$	0
(3.18)	$\sigma^{a'}_{b'c'}$	0	$-\frac{2}{3} R^a_{(\alpha b c)} u^\alpha$
(3.19)	$\sigma^a_{b'c'}$	0	$\frac{1}{2} R^a_{b\alpha c} u^\alpha - \frac{1}{3} R^a_{(\alpha b c)} u^\alpha$
(3.20)	$\sigma^{a'}_{b'c'd'}$	$-\frac{2}{3} R^a_{(c b d)}$	$\frac{1}{2} R^a_{(c b d;\alpha)} u^\alpha - \frac{2}{3} R^a_{(\alpha b d);c} u^\alpha - \frac{2}{3} R^a_{(\alpha b c);d} u^\alpha$
(3.21)	$\sigma^a_{b'c'd'}$	$-\frac{1}{3} R^a_{(c b d)} - \frac{1}{2} R^a_{bcd}$	$-\frac{1}{2} R^a_{(c b d;\alpha)} u^\alpha + \frac{2}{3} R^a_{(\alpha b d);c} u^\alpha$
(3.22)	$g_{a'}^b$	g_a^b	0
(3.24)	$g_a^{b'}_{;c'}$	0	$\frac{1}{2} R^b_{a\alpha c} u^\alpha$
(3.29)	$g_a^{b'}_{;c'd'}$	$-\frac{1}{2} R^b_{acd}$	$-\frac{2}{3} R^b_{ac(d;\alpha)} u^\alpha$
(3.34)	$\Delta^{1/2}$	1	0
(3.46)	V_0	$\frac{1}{2} m^2 + \frac{1}{2} (\xi - \frac{1}{6}) R$	$-\frac{1}{4} (\xi - \frac{1}{6}) R_{;\alpha} u^\alpha$

TABLE III: Initial conditions for tensors used in the numerical calculation of V_0 .

B. Results

The accuracy of our numerical code may be verified by comparing with the results of Ref. [43], which gives analytic expressions for all of the bi-tensors used in this paper. In Figs. 1 and 2, we compare analytic and numerical expressions for $\Delta^{1/2}$ and V_0 , respectively. We consider the null geodesic which starts at $\rho = 0.5$ and moves inwards to $\rho = 0.25$ before turning around and going out to $\rho = 0.789$, where it reaches a caustic (i.e. the edge of the normal neighborhood). The affine parameter, s , has been scaled so that it is equal to the angle coordinate, ϕ . We find that the numerical results faithfully match the analytic solution up to the boundary of the normal neighborhood. For the case of $\Delta^{1/2}$, Fig. 1, the error remains less than one part in 10^{-6} to within a short distance of the normal neighborhood boundary. The results for $V_0(x, x')$ are less accurate, but nonetheless the relative error remains less 1%.

In Fig. 3, we give plot calculated from our numerical code which indicates how $\Delta^{1/2}$ varies over the whole light-cone in Schwarzschild. We find that it remains close to its initial value of 1 far away from the caustic. As geodesics get close to the caustic, $\Delta^{1/2}$ grows and is eventually singular at the caustic. This is exactly as expected; $\Delta^{1/2}$ is a measure

Equation	Terms involving $(s')^{-1}$	Initial condition for $(s')^{-1}$ terms
(3.11)	$-(s')^{-1} \left(\xi^{\alpha'}_{\alpha'} \xi^{\alpha'}_{b'} - \xi^{\alpha'}_{b'} \right)$	0
(3.13)	$-(s')^{-1} \left(\eta^a_{\alpha'} \xi^{\alpha'}_{b'} - \eta^a_{b'} \right)$	0
(3.18)	$(s')^{-1} \left(\sigma^{\alpha'}_{b'c'} - \xi^{\alpha'}_{\alpha'} \sigma^{\alpha'}_{b'c'} - \xi^{\alpha'}_{b'} \sigma^{\alpha'}_{\alpha'c'} - \xi^{\alpha'}_{c'} \sigma^{\alpha'}_{\alpha'b'} \right)$	$-\frac{2}{3} R^a_{(b \alpha c)} u^\alpha$
(3.19)	$(s')^{-1} \left(\sigma^a_{b'c'} - \eta^a_{\alpha'} \sigma^{\alpha'}_{b'c'} - \xi^{\alpha'}_{b'} \sigma^a_{\alpha'c'} - \xi^{\alpha'}_{c'} \sigma^a_{\alpha'b'} \right)$	$-\frac{1}{2} R^a_{abc} u^\alpha - \frac{1}{3} R^a_{(b \alpha c)} u^\alpha$
(3.20)	$(s')^{-1} \left(\sigma^{\alpha'}_{b'c'd'} - \sigma^{\alpha'}_{\alpha'b'} \sigma^{\alpha'}_{c'd'} - \sigma^{\alpha'}_{\alpha'c'} \sigma^{\alpha'}_{b'd'} - \sigma^{\alpha'}_{\alpha'd'} \sigma^{\alpha'}_{b'c'} \right. \\ \left. - \sigma^{\alpha'}_{\alpha'b'c'} \xi^{\alpha'}_{d'} - \sigma^{\alpha'}_{\alpha'b'd'} \xi^{\alpha'}_{c'} - \sigma^{\alpha'}_{\alpha'c'd'} \xi^{\alpha'}_{b'} - \sigma^{\alpha'}_{b'c'd'} \xi^{\alpha'}_{\alpha'} \right)$	$-\frac{3}{2} R^a_{(b \alpha c;d)} u^\alpha$
(3.21)	$(s')^{-1} \left(\sigma^a_{b'c'd'} - \sigma^a_{\alpha'b'} \sigma^{\alpha'}_{c'd'} - \sigma^a_{\alpha'c'} \sigma^{\alpha'}_{b'd'} - \sigma^a_{\alpha'd'} \sigma^{\alpha'}_{b'c'} \right. \\ \left. - \sigma^a_{\alpha'b'c'} \xi^{\alpha'}_{d'} - \sigma^a_{\alpha'b'd'} \xi^{\alpha'}_{c'} - \sigma^a_{\alpha'c'd'} \xi^{\alpha'}_{b'} - \sigma^{\alpha'}_{b'c'd'} \eta^a_{\alpha'} \right)$	$\frac{1}{2} R^a_{(c \alpha d);b} u^\alpha - \frac{5}{6} R^a_{b\alpha(c;d)} u^\alpha + \frac{7}{6} R^a_{(d \alpha b ;c)} u^\alpha$
(3.22)	0	0
(3.24)	$-(s')^{-1} g^b_{a'} \xi^{\alpha'}_{c'}$	$-\frac{1}{2} R^b_{aac} u^\alpha$
(3.29)	$-(s')^{-1} \left(g^b_{a'} \xi^{\alpha'}_{c'} + g^b_{a'} \xi^{\alpha'}_{d'} + g^b_{a'} \sigma^{\alpha'}_{c'd'} \right)$	$-\frac{2}{3} R^b_{a\alpha(c;d)} u^\alpha$
(3.34)	$-(s')^{-1} \Delta^{1/2} \left(\xi^{\alpha'}_{a'} - \delta^{\alpha'}_{a'} \right)$	0
(3.46)	$-(s')^{-1} \left[\left(\xi^{\alpha'}_{a'} - \delta^{\alpha'}_{a'} \right) V_0 + 2V_0 + (\square' - m^2 - \xi R) \Delta^{1/2} \right]$	$\frac{1}{4} \left(\xi - \frac{1}{6} \right) R_{;\alpha} u^\alpha$

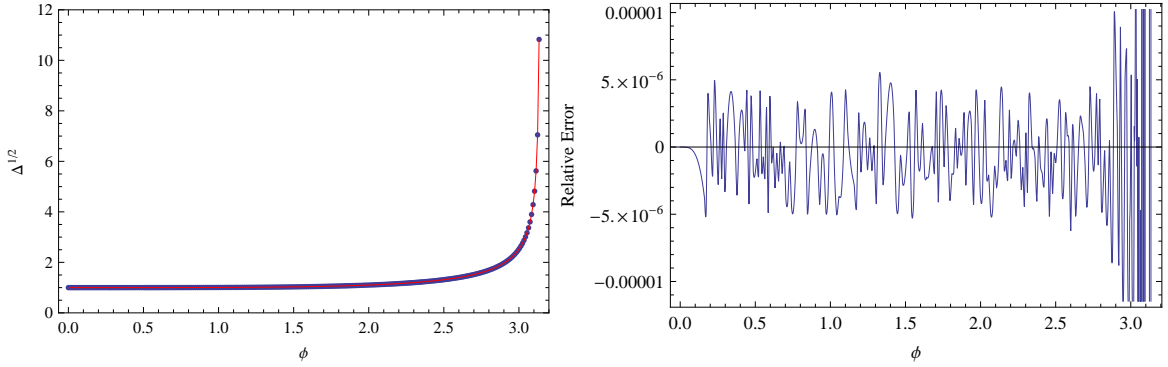
TABLE IV: Initial conditions for transport equations required for the numerical calculation of V_0 .

FIG. 1: Comparison of numerical and exact analytic calculations of $\Delta^{1/2}$ as a function of the angle, ϕ , along an orbiting null geodesic in Nariai spacetime. Left: The numerical calculation (blue dots) is a close match with the analytic expression (red line). Right: With parameters so that the code completes in 1 minute, the relative error is within 0.0001% up to the boundary of the normal neighborhood (at $\phi = \pi$).

of the strength of focusing of geodesics, where values greater than 1 correspond to focusing and values less than 1 correspond to de-focusing. At the caustic, geodesics are focused to a point one would expect $\Delta^{1/2}$ to be singular there.

In Fig. 4, we give a similar plot (again calculated from our numerical code) which indicates how $V(x, x')$ varies over the light-cone in Schwarzschild. In this case there is considerably more structure than was previously the case with $\Delta^{1/2}$. There is the expected singularity at the caustic. However, travelling along a geodesic, $V(x, x')$ also becomes negative for a period before turning positive and eventually becoming singular at the caustic.

The transport equations may also be applied to calculate $V_r(x, x')$ along a timelike geodesic. In Fig. 5, we apply our numerical code to the calculation of $V_0(x, x')$ along the timelike circular orbit at $r = 10M$ in Schwarzschild.

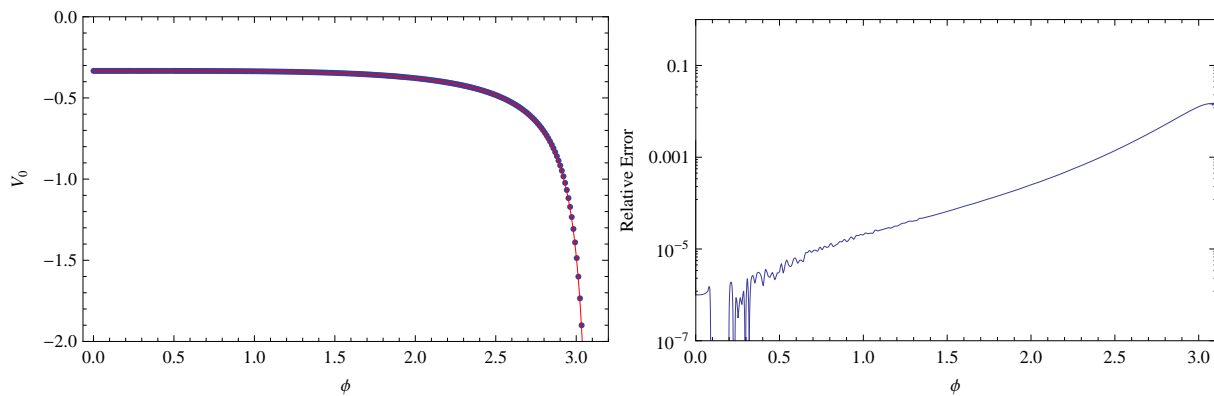


FIG. 2: Comparison of numerical and exact analytic calculations of V_0 for a massless, minimally coupled scalar field as a function of the angle, ϕ , along an orbiting null geodesic in Nariai spacetime. Left: The numerical calculation (blue dots) is a close match with the analytic expression (red line). The coincidence value is $V(x, x) = \frac{1}{2}(\xi - \frac{1}{6})R = -\frac{1}{3}$, as expected. Right: With parameters so that the code completes in 1 minute, the relative error in the numerical calculation is within 1% up to the boundary of the normal neighborhood (at $\phi = \pi$),

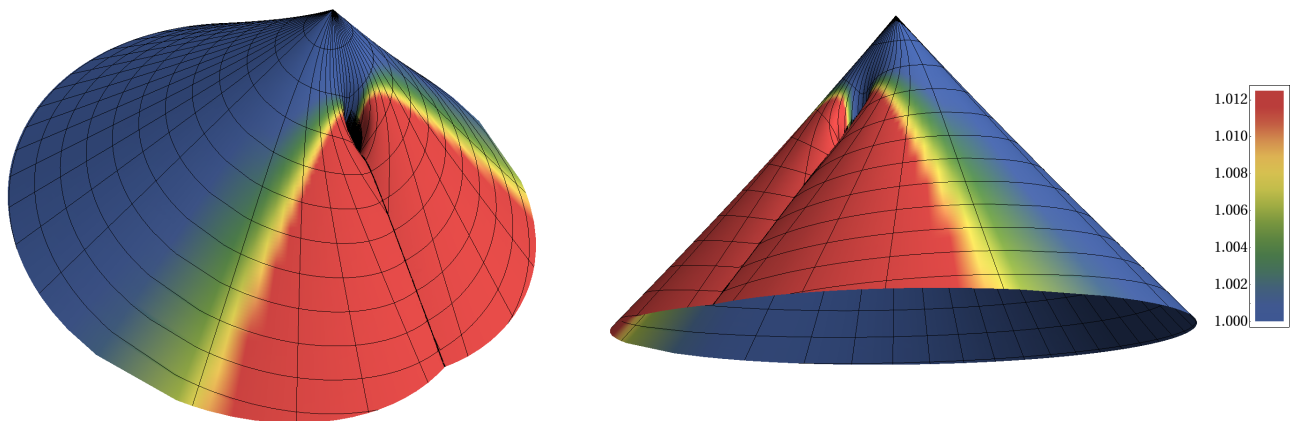


FIG. 3: (Colour online) $\Delta^{1/2}$ along the light-cone in Schwarzschild Spacetime. The point x at the vertex of the cone is fixed at $r = 10M$. $\Delta^{1/2}$ increases along a geodesic up to the caustic where it is singular.

VI. DISCUSSION

Several of the covariant expansion expressions computed by our code using the Avramidi method have been previously given in Ref. [18], albeit to considerably lower order (for example, Ref. [18] gives $V(x, x')$ to order $(\sigma^a)^4$ compared to order $(\sigma^a)^{20}$ here). Comparison between the two results gives exact agreement, providing a reassuring confirmation of the accuracy of both our expressions and those of Ref. [18]. Furthermore, several of the expansions not given by Decanini and Folacci may be compared with those found by Christensen [12, 13]. Again, we have found that our code is in exact agreement with Christensen's results.

In Sec. V, we discussed a numerical implementation of the transport equation approach to the calculation of V_0 . This implementation is capable of computing V_0 for any spacetime, although we have chosen Nariai and Schwarzschild spacetimes as examples. The choice of Nariai spacetime has the benefit that an expression for V_0 is known exactly [43]. This makes it possible to compare our numerical results with the analytic expressions to determine both the validity of the approach and the accuracy of the numerical calculation. Given parameters allowing the code to run in under a minute, we find that the numerical implementation is accurate to less than 1% out as far as the location of the singularity of V_0 , at the caustic point (i.e. everywhere within the normal neighborhood).

Our *Mathematica* implementation of the semi-recursive approach to covariant expansions (Sec. IV) is given as a

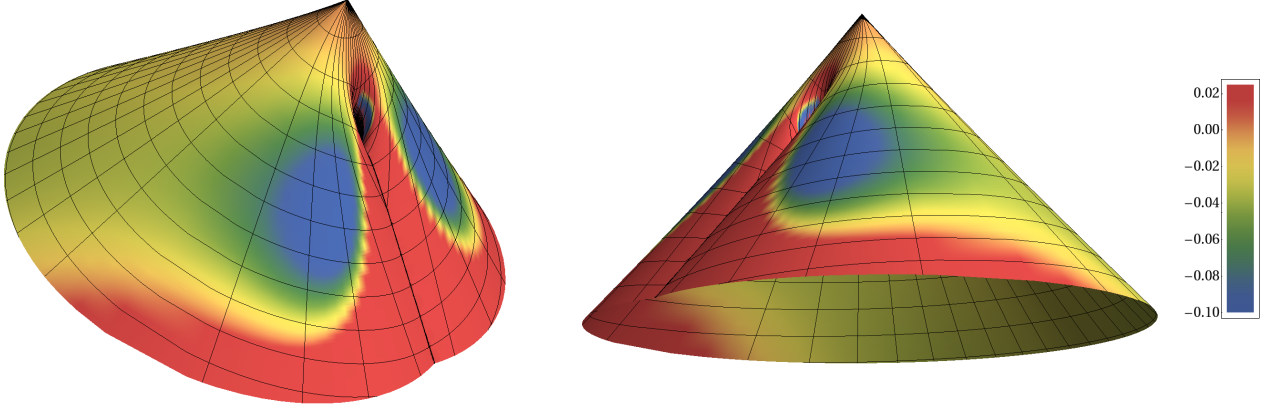


FIG. 4: (Colour online) $V(x, x')$ for a massless, scalar field along the light-cone in Schwarzschild Spacetime. The point x (the vertex of the cone) is fixed at $r = 10M$. $V(x, x')$ is 0 initially, then, travelling along a geodesic, it goes negative for a period before turning positive and eventually becoming singular at the caustic.

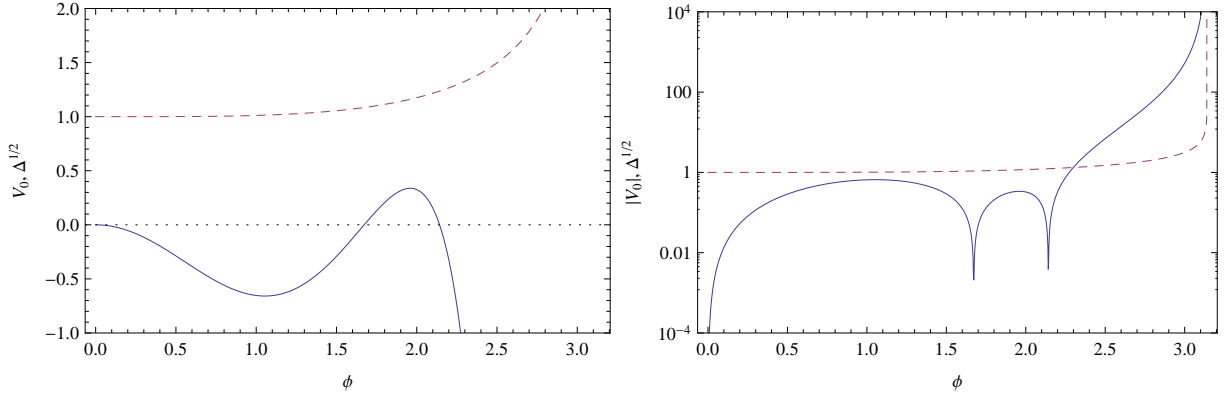


FIG. 5: (Colour online) $V_0(x, x')$ (solid blue line) and $\Delta^{1/2}$ (dashed purple line) for a massless, scalar field along the timelike circular orbit at $r = 10M$ in Schwarzschild Spacetime as a function of the angle, ϕ through which the geodesic has passed. In the logarithmic plot, the absolute value of $V_0(x, x')$ is plotted, since $V_0(x, x')$ is negative between $\phi \sim 1.7$ and $\phi \sim 2.2$. The apparent discontinuities in the plot are therefore simply a result of $V_0(x, x')$ passing through 0 at these points.

proof of principle. We believe than considerable improvement could be achieved, particularly in the limiting area of memory requirements. The expressions for the HaMiDeW coefficients as computed by our code are very general. However, they are not necessarily given as a minimal set. For example, the DeWitt coefficient a_3 may be written as a sum of four terms, yet our code produces a sum of seven equivalent terms. It is quite possible, however, that a set of transformation rules could be produced to reduce our expression to a canonical form such as that of Ref. [15]. In fact, some progress in that direction has been made already with the *Invar* [45, 46] software package, which is able to quickly canonicalize scalar invariants. As our code is already written in *Mathematica* [47] and has the ability to output into the *xTensor* [23] notation used by *Invar* [45, 46], an extension of the package to allow for the canonicalization of tensor invariants would allow our expressions to be immediately canonicalized with no further effort.

In integrating the transport equations given in Sec. III along a specific geodesic, we are not limited to the normal neighborhood. The only difficulty arises at *caustics*, where some bi-tensors such as $\Delta^{1/2}$ and V_0 become singular. However, this is not an insurmountable problem. The singular components may be separated out and methods of complex analysis employed to integrate through the caustics, beyond which the bi-tensors once more become regular (but not necessarily real-valued) [2]. This is highlighted in Fig. 5, where our plot of $\Delta^{1/2}$ and V_0 extends outside the normal neighborhood, the boundary of which is at $\phi \approx 1.25$, where the first null geodesic re-intersects the orbit. It does not necessarily follow, however, that the Green function outside the normal neighborhood is given by this value for $V(x, x')$. Instead, one might expect to obtain the Green function by considering the sum of the contributions

obtained by integrating along *all* geodesics connecting x and x' (there will be a discrete number of such geodesics except at caustics).

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